Second Order Response Examples

Example 2

1. **Determine the un-damped natural frequency and damping ratio of the following second order system.**

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

2. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time 2% and 5% criterion t_s

Answer

1. **Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.**

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \quad \Rightarrow \omega_n = 2 \; rad/\sec$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 4$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = \frac{1}{\omega_n} = \frac{1}{2}$$

$$\Rightarrow \zeta = 0.5$$

1. Rise Time.

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

$$t_r = \frac{3.14 - \tan^{-1}\frac{\sqrt{1-(0.5)^2}}{0.5}}{2\sqrt{1-(0.5)^2}}$$

$$t_r = \frac{3.14 - \tan^{-1}\frac{\sqrt{0.75}}{0.5}}{2 * \sqrt{1-(0.5)^2}}$$

$$t_r = \frac{3.14 - \tan^{-1}(1.732)}{2 * \sqrt{0.75}}$$

$$t_r = \frac{3.14 - 1.05}{2 * 0.866} = \frac{2.09}{1.72} = 1.215$$

2. Peak Time.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$t_p = \frac{3.14}{2 * \sqrt{1-(0.5)^2}} = \frac{3.14}{2 * \sqrt{0.75}} = \frac{3.14}{1.72} = 1.8255$$

3. Maximum Peak

$$M_P = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

$$M_P = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

$$M_P = e^{-\frac{3.14 * 0.5}{\sqrt{1-(0.5)^2}}}$$

$$M_P = e^{-\frac{1.57}{\sqrt{0.75}}} = e^{-\frac{1.57}{0.866}} = e^{-1.8129} = 0.9054$$

4. Settling Time.

Settling time 2% $$t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.5 * 2} = \frac{4}{1} = 4$$

Settling time 5% $$t_s = \frac{3}{\xi\omega_n} = \frac{3}{0.5 * 2} = \frac{3}{1} = 3$$

Example 3

For the second order system described by the closed loop transfer function T(s),

1. find ω_n and ξ

$$T(s) = \frac{C(s)}{R(s)} = \frac{6}{s^2 + 14s + 64}$$

2. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time 2% and 5% criterion t_s

Answer

Compare with the standard equation we have

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 64 \quad ==> \quad \omega_n = 8$$

$$2\zeta\omega_n = 14 \quad ==> \quad \zeta = \frac{14}{2\omega_n} = \frac{14}{16} = 0.8$$

$$k\omega_n = 6 \quad ==> \quad k = \frac{6}{8} = 0.75$$

1. Rise Time.

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

$$t_r = \frac{3.14 - \tan^{-1}\dfrac{\sqrt{1-(0.8)^2}}{0.8}}{8 * \sqrt{1-(0.8)^2}}$$

$$t_r = \frac{3.14 - \tan^{-1}\dfrac{\sqrt{0.36}}{0.8}}{8 * \sqrt{0.36}}$$

$$t_r = \frac{3.14 - \tan^{-1}(0.75)}{8 * 0.6}$$

$$t_r = \frac{3.14 - 0.64}{4.2} = \frac{2.5}{4.2} = 0.595$$

2. Peak Time.

$$t_p = \frac{3.14}{8 * \sqrt{1-(0.8)^2}} = \frac{3.14}{8 * \sqrt{0.36}} = \frac{3.14}{8 * 0.6} = \frac{3.14}{4.2} = 0.7476$$

3. Maximum Peak

$$M_P = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

$$M_P = e^{-\frac{3.14 * 0.8}{\sqrt{1-(0.8)^2}}}$$

$$M_P = e^{-\frac{2.512}{\sqrt{0.36}}} = e^{-\frac{2.512}{0.6}} = e^{-4.186} = -1.467$$

4. Settling time

Settling time 2%

$$t_s = \frac{4}{\xi\omega_n} = \frac{4}{8 * 0.8} = \frac{4}{6.4} = 0.625$$

Settling time 5%

$$t_s = \frac{3}{\xi\omega_n} = \frac{3}{8 * 0.8} = \frac{3}{6.4} = 0.4687$$

Example 4

1. **Determine the un-damped natural frequency and damping ratio of the following second order system.**

$$T(s) = \frac{20}{s^2 + 6s + 144}$$

2. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time 2% and 5% criterion t_s

Answer

Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

1.

ω_n^2 and ξ

$\omega_n^2 = 144 \rightarrow \omega_n = 12$

$2\omega_n \xi = 6 \rightarrow \xi = \frac{6}{2\omega n} = \frac{6}{24} = 0.25$

$$\omega n\, k = 20 \quad ==> \quad k = \frac{20}{12} = 1.66$$

2. Home work

Home work

For the following transfer function

$$T(s) = \frac{5}{(s+3)(s+6)}$$

1. **Determine the un-damped natural frequency and damping ratio of the following second order system.**

2. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time 2% and 5% criterion t_s

Answer

$$\mathbf{T(s) = \frac{5}{(s+3)(s+6)}}$$

$$\mathbf{T(s) = \frac{5}{s^2 + 6s + 3s + 18}}$$

$$\mathbf{T(s) = \frac{5}{s^2 + 9s + 18}}$$