



Stability

Learning Outcomes

After completing this lecture the student will be able to:

- Make and interpret a basic Routh table to determine the stability of a system.
- Make and interpret a Routh table where either the first element of a row is zero or an entire row is zero.

Stability

❖ **Stable systems**

have closed-loop transfer functions with poles in the left half-plane.

❖ **Unstable systems**

have closed-loop transfer functions with at least one pole in the right half-plane, and/or poles of multiplicity greater than one on the imaginary axis

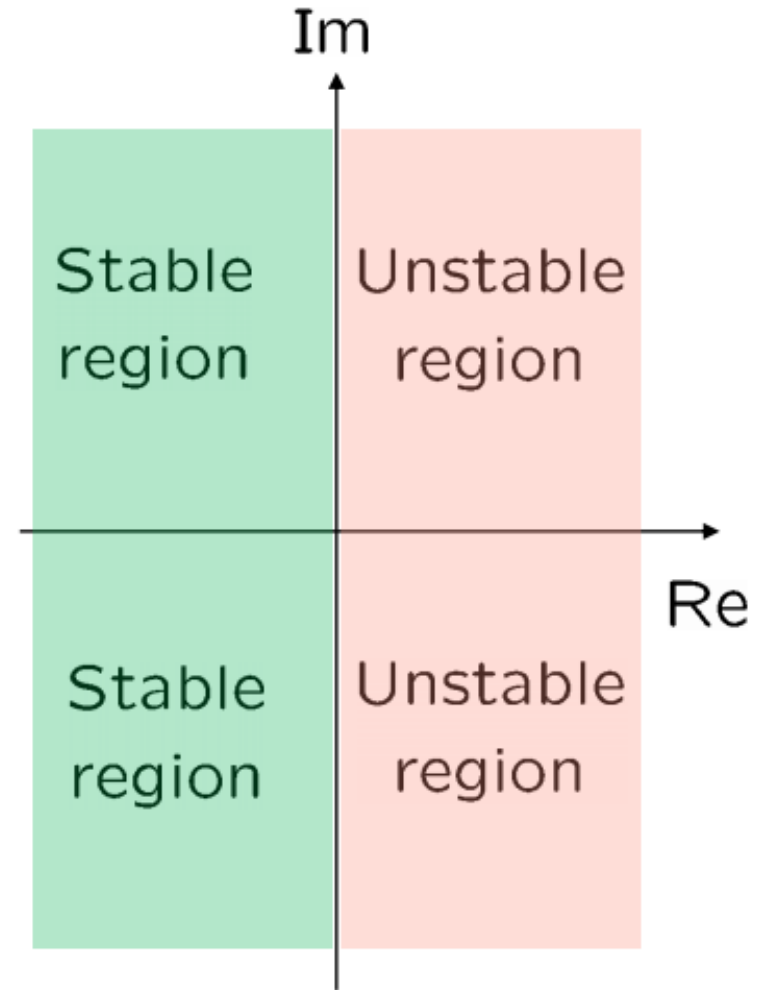
❖ **Marginally stable systems**

have closed-loop transfer functions with only imaginary axis poles of multiplicity one and poles in the left half-plane.

How to Determine System Stability?

Let us focus on the natural response definitions of stability

- Stable systems have closed-loop transfer functions with poles only on the left half-plane.
- If the closed-loop system poles are on the right half of the s-plane and hence have a positive real part, the system is unstable.



Stability Determined by TF Poles

$G(s)$	Stable	marginally stable	unstable
$\frac{5(s+2)}{(s+1)(s^2+s+1)}$?	
$\frac{5(-s+2)}{(s+1)(s^2+s+1)}$?	
$\frac{5}{(s-2)(s^2+3)}$?	
$\frac{s^2+2}{(s+1)(s^2-s+1)}$?	
$\frac{5}{(s+2)(s^2+1)^2}$?	
$\frac{1}{(s^2-1)(s+1)}$?	

Routh-Hurwitz Stability Criterion

- It is a method for determining continuous system stability.
- The Routh-Hurwitz criterion states that “*the number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the Routh array*”.

Advantages of Routh-Hurwitz Criterion

1. We can find the stability of the system without solving the equation.
2. We can easily determine the relative stability of the system.
3. we can determine the range of K for stability.
4. we can also determine the point of intersection for root locus with an imaginary axis.

Disadvantages of Routh-Hurwitz Criterion

1. This criterion is applicable only for a linear system.
2. It does not provide the exact location of poles on the right and left half of the S plane.
3. The characteristic equation, it is valid only for real coefficients.

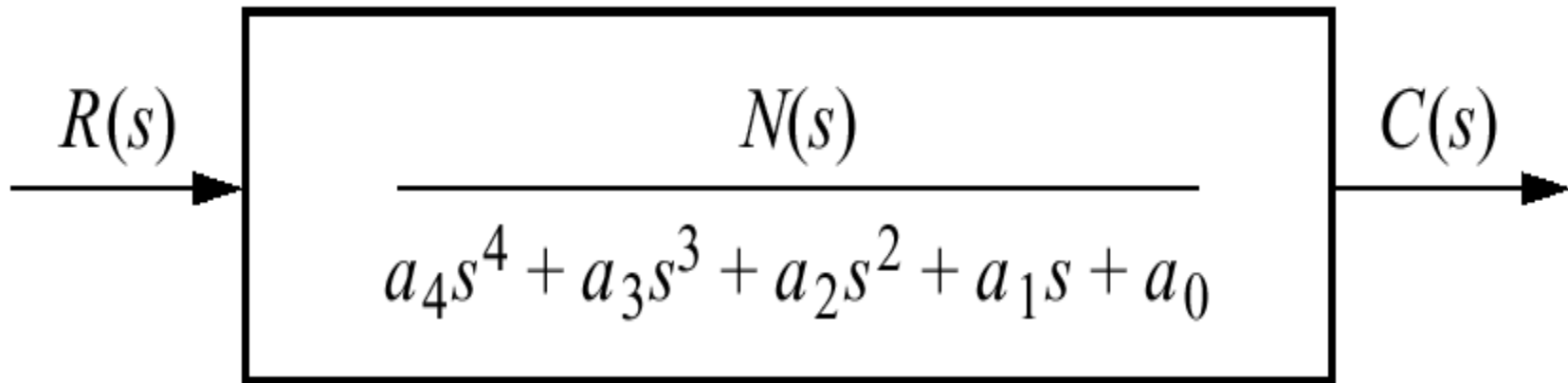
Routh-Hurwitz Stability Criterion

- This method yields stability information without the need to solve for the closed-loop system poles.

The method requires to Generate a data table called a Routh table.

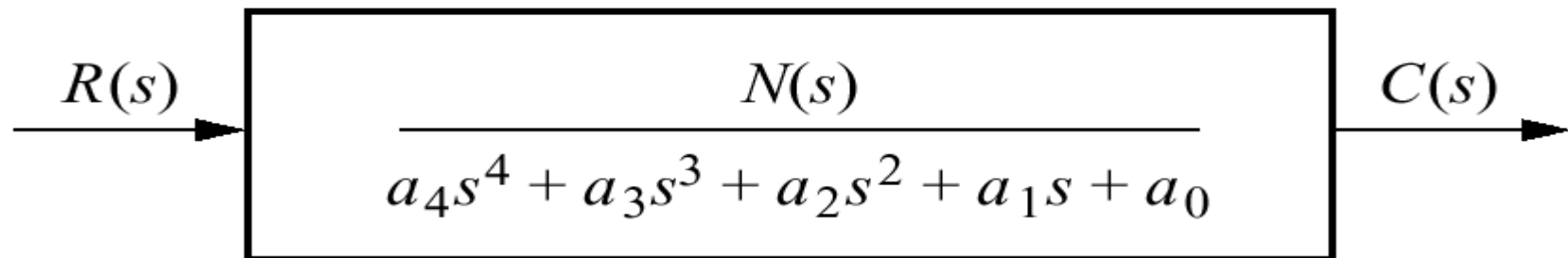
1. Arrange all the coefficients of the above equation in two rows:
2. From these two rows we will form the third row
3. Now, we shall form fourth row by using second and third row:
4. We shall continue this procedure of forming a new rows:

Example: Generating a basic Routh Table.



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Example: Generating a basic Routh Table.



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$b_1 = \frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3}$	$b_2 = \frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3}$	b3 = $\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3}$
s^1	$c_1 = \frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1}$	c2 = $\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1}$	c3 = $\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1}$
s^0	$d_1 = \frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1}$	d2 = $\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1}$	d3 = $\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1}$

0

0

Generating a Basic Routh Table

- First label the rows with powers of s from highest power of s down to lowest power of s in a vertical column.
- Next form the first row of the Routh table, using the coefficients of the denominator of the closed-loop transfer function (characteristic equation).
- Start with the coefficient of the highest power and skip every other power of s .
- Now form the second row with the coefficients of the denominator skipped in the previous step.
- The table is continued horizontally and vertically until zeros are obtained.
- For convenience, any row can be multiplied or divide by a positive constant before the next row is computed without changing the values of the rows below and disturbing the properties of the Routh table.

Routh's Stability Condition

- If the closed-loop transfer function has all poles in the left half of the s -plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table.
- The Routh-Hurwitz criterion declares that the number of roots of the polynomial that are lies in the right half-plane is equal to the number of sign changes in the first column. Hence the system is unstable if the poles lies on the right hand side of the s -plane.

Four Special Cases or Configurations in the First Column Array of the Routh's Table:

1. **Case-I:** No element in the first column is zero.
2. **Case-II:** A zero in the first column but some other elements of the row containing the zero in the first column are nonzero.
3. **Case-III:** A zero in the first column and the other elements of the row containing the zero are also zero.
4. **Case-IV:** As in the third case but with repeated roots on the $j\omega$ -axis.

Case-I: No element in the first column is zero.

Second-Order System.

The characteristic polynomial of a second order system is given below

$$q(s) = a_2s^2 + a_1s + a_0$$

The Routh array is written as

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & b_1 & 0 \end{array}$$

Where

$$b_1 = \frac{a_1a_0 - (0)a_2}{a_1} = \frac{-1}{a_1} \begin{vmatrix} a_2 & a_0 \\ a_1 & 0 \end{vmatrix} = a_0$$

The requirement for a stable second order system is simply that all the coefficient be positive or all the coefficient s be negative.

Third-Order System.

The characteristic polynomial of a third order system is given below

$$q(s) = a_3s^3 + a_2s^2 + a_1s + a_0$$

The Routh array is

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b_1 & 0 \\ s^0 & c_1 & 0 \end{array}$$

Where

$$b_1 = \frac{a_2a_1 - a_0a_3}{a_2} \quad \text{and} \quad c_1 = \frac{b_1a_0}{b_1} = a_0$$

- The requirement for a stable third order system is that the coefficients be positive and $a_2a_1 > a_0a_3$.
- The condition when $a_2a_1 = a_0a_3$ results in a marginally stability case (recognized as Case-3 because there is a zero in the first column) and one pair of roots lies on the imaginary axis in the s-plane.

Example-1: Find the stability of the continuous system having the characteristic equation of

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

$$b_1 = \frac{2 * 3 - 1 * 4}{2} = 1$$

▪ Routh table:

s^4	1	3	5
s^3	2	4	
s^2			
s^1			
s^0			

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

▪ Routh table:

s^4	1	3	5
s^3	2	4	
s^2	1		
s^1			
s^0			

$$b_1 = \frac{2 * 3 - 1 * 4}{2} = 1$$

$$b_2 = \frac{2 * 5 - 1 * 0}{2} = 5$$

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

▪ Routh table:

s^4	1	3	5
s^3	2	4	
s^2	1	5	
s^1			
s^0			

$$b_1 = \frac{2 * 3 - 1 * 4}{2} = 1$$

$$b_2 = \frac{2 * 5 - 1 * 0}{2} = 5$$

$$c_1 = \frac{1 * 4 - 5 * 2}{1} = -6$$

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

▪ Routh table:

s^4	1	3	5	+
s^3	2	4		+
s^2	1	5		+
s^1	-6			-
s^0	5			+

Sign

$$b_1 = \frac{2 * 3 - 1 * 4}{2} = 1$$

$$b_2 = \frac{2 * 5 - 1 * 0}{2} = 5$$

$$c_1 = \frac{1 * 4 - 5 * 2}{1} = -6$$

$$d_1 = \frac{-3 * 5 - 1 * 0}{-3} = 5$$

Example-2: Find the stability of the continuous system having the characteristic equation of

$$s^3 + 6s^2 + 12s + 8 = 0$$

The Routh table of the given system is computed and shown is the table below;

s^3	1	12	0
s^2	6	8	0
s^1	$\frac{64}{6}$	0	
s^0	8		

- Since there is **no changes of the sign** in the first column of the Routh table, it means that all the roots of the characteristic equation have negative real parts and **hence this system is stable**.

Example-3: Find the stability of the continuous system having the characteristic polynomial of a third order system is given below

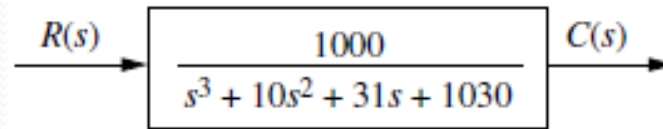
$$s^3 + s^2 + 2s + 24$$

- The Routh array is

s^3		1	2
s^2		1	24
s^1		-22	0
s^0		24	0

- Because **TWO changes in sign** appear in the first column, we find that two roots of the characteristic equation lie in the right hand side of the s-plane.
Hence the system is unstable.

Example-4: Find the stability of the system shown below using Routh criterion.



The close loop transfer function is shown in the figure

The Routh table of the system is shown in the table

s^3	1	31	0
s^2	10	1030	0
s^1	$\frac{\begin{vmatrix} 1 & 31 \\ 10 & 1030 \end{vmatrix}}{1} = -72$	$\frac{\begin{vmatrix} 1 & 0 \\ 10 & 0 \end{vmatrix}}{1} = 0$	$\frac{\begin{vmatrix} 1 & 0 \\ 10 & 0 \end{vmatrix}}{1} = 0$
s^0	$\frac{\begin{vmatrix} 1 & 1030 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\frac{\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\frac{\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

Because **TWO changes in sign** appear in the first column, we find that two roots of the characteristic equation lie in the right hand side of the s-plane. **Hence the system is unstable.**

Example-5: Find the stability of the system shown below using Routh criterion.

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

- The Routh table of the system is

s^4	2	3	10
s^3	1	5	0
s^2	$\frac{(1)(3) - (2)(5)}{1} = -7$	10	0
s^1	$\frac{(-7)(5) - (1)(10)}{-7} = 6.43$	0	0
s^0	10	0	0

Example-5: Find the stability of the system shown below using Routh criterion.

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

- The Routh table of the system is

s^4	2	3	10
s^3	1	5	0
s^2	-7	10	0
s^1	6.43	0	0
s^0	10	0	0

- System is unstable** because there are **two sign changes** in the first column of the Routh's table. Hence the equation has two roots on the right half of the s-plane.

Example-6: Determine the stability of the closed-loop transfer function;

$$D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$$

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	-1.75		
s^0	1		

- Since there are **TWO sign changes**, the **system is unstable** and has **TWO right-half-plane poles**.
- This is the same as the result obtained in the previous Example.
- Notice that Table does not have a zero in the first column.

Example-7: Determine the stability of the closed-loop transfer function;

$$Q(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$$

Routh array

s^5	1	2	1
s^4	1	2	1
s^3	0 4	0 4	
s^2	1	1	
s^1	0 2		
s^0	1		

Derivative of auxiliary poly.

$$(s^4 + 2s^2 + 1)' = 4s^3 + 4s$$

$$(s^2 + 1)' = 2s$$

No sign changes
in the first column



No root in OPEN RHP